Multi-period Stochastic Optimization Models for Dynamic Asset Allocation

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Abstract

Asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns subject to various uncertainties, policy and legal constraints, and other requirements. In order to determine the asset mix explicitly, one may use a multi-period portfolio optimization model.

The concept of scenarios is typically employed for modeling random parameters in multi-period stochastic programming (MSP) models, and scenarios are constructed via a tree structure. Another approach for developing dynamic investment strategies, which offers an alternative to stochastic programming, is the dynamic stochastic control. Recently, an alternative stochastic programming model with simulated paths was proposed by Hibiki (2001b, 2001c). Hibiki(2003) developed the general formulation for several investment strategies, and highlight its features and properties using some numerical tests. Scenarios are constructed via a simulated path structure. The advantage of simulated paths comparing to scenario trees is higher accuracy of description of uncertainties associated with asset returns. In addition, we can make conditional decisions in this framework similarly to a scenario tree model. This model is called a hybrid model. It can be easily implemented and efficiently solved using sophisticated mathematical programming software.

In this paper, we compare two types of multi-period stochastic optimization (MPSO) models, and clarify that the hybrid model can evaluate and control risk better than the scenario tree model using some numerical tests. According to the numerical results, the efficient frontier of the hybrid model with the fixed-proportion strategy dominates that of the scenario tree model when we evaluate them on the simulated paths. Moreover, the optimal solutions of the hybrid model is more appropriate than those of the scenario tree model, which are very extream ones.

1   Introduction

Rational investors maximize the expected utility of return from their investment portfolio, or minimize the risk exposure of return, subject to the required expected return. They must decide on their optimal portfolio in securities in order to meet their satisfaction. This paper discusses optimal dynamic investment policies for investors, who make the investment decision in each asset category over time. This problem is called "dynamic asset allocation".

Asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns, in presence of uncertainties and subject to various legal constraints, policies, and other requirements. In order to determine the asset mix explicitly, a multi-period portfolio optimization model can be used.

It is critical for stochastic modeling to handle uncertainties and investment decisions appropriately. The decisions have to be independent from knowledge of actual paths that will occur. Thus, we must define a set of decision variables and a set of constraints to prevent the optimization model from being solved by anticipating the event in the future. In addition, we need a sufficient number of paths to get a better accuracy with respect to the future possible events.

The concept of scenarios is typically employed for modeling random parameters in multi-period stochastic programming (MSP) models. Scenarios are constructed via a tree structure (see Mulvey and Ziemba, 1995 and 1998 for a detail discussion). This model is based on the expansion of the decision space, taking into account the conditional nature of the scenario tree. Conditional decisions are made at each node, subject to the modeling constraints. To ensure that the constructed representative set of scenarios covers the set of possibilities to a sufficient degree, the number of decision variables and constraints in the scenario tree may grow exponentially. This model is called a scenario tree model.

Recently, an alternative stochastic programming model using simulated paths was proposed by Hibiki (2001b). Hibiki(2003) developed the general formulation for several investment strategies, and highlight its features and properties by using some numerical tests. Scenarios are constructed via a simulated path structure. We can generate sample paths associated with asset returns using the Monte Carlo simulation method. Therefore, the advantage of simulated paths compared to a scenario tree gives a better descriptive accuracy of the uncertainties associated with asset returns. In addition, we can make conditional decisions in this framework as with a scenario tree model. This model is called a "hybrid model", because it allows for better accuracy in describing the uncertainties as well as for conditional decisions [1] . It can be easily implemented and efficiently solved using sophisticated mathematical programming software.

The hybrid model is developed to overcome the shortcoming of the scenario tree model associ-

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[1]Hibiki(2000 and 2001a) developed the simulated path model. The model also requires simulated paths to have the accuracy of uncertainties, but it cannot make conditional decisions. The hybrid model is allowed to expand the decision space and to make conditional decisions as used in the scenario tree model. The simulated path model is a special version of the hybrid model.

ated with the uncertainties. Therefore, it is important to answer the question how quantitatively the hybrid model is better than the scenario tree model, which was not shown in the previous papers(Hibiki, 2001b and 2003). In this paper, we compare two types of multi-period stochastic optimization (MPSO) models, and clarify that the hybrid model can evaluate and control risk better than the scenario tree model by using some numerical tests.

We need the following developments to solve this problem. At first, we develop the iterative algorithm to solve the hybrid model with the fixed-proportion strategy, which is formulated as the non-convex program. This is because two kinds of models should be compared using the same strategy. Second, we propose the procedure of comparing them in the simulated path framework.

The paper is organized as follows. Section 2 presents the concept and formulations of two kinds of models, and develops the iterative algorithm to solve the hybrid model with the fixed-proportion strategy. In Section 3, we demonstrate the scenario generating process and the procedure of generating the extended decision tree, and explain how to generate a scenario tree from simulated paths to compare the scenario tree model with the hybrid model. Section 4 presents some numerical tests for various cases. Section 5 provides some concluding remarks and outlines our future research.

2   Multi-period stochastic programming models

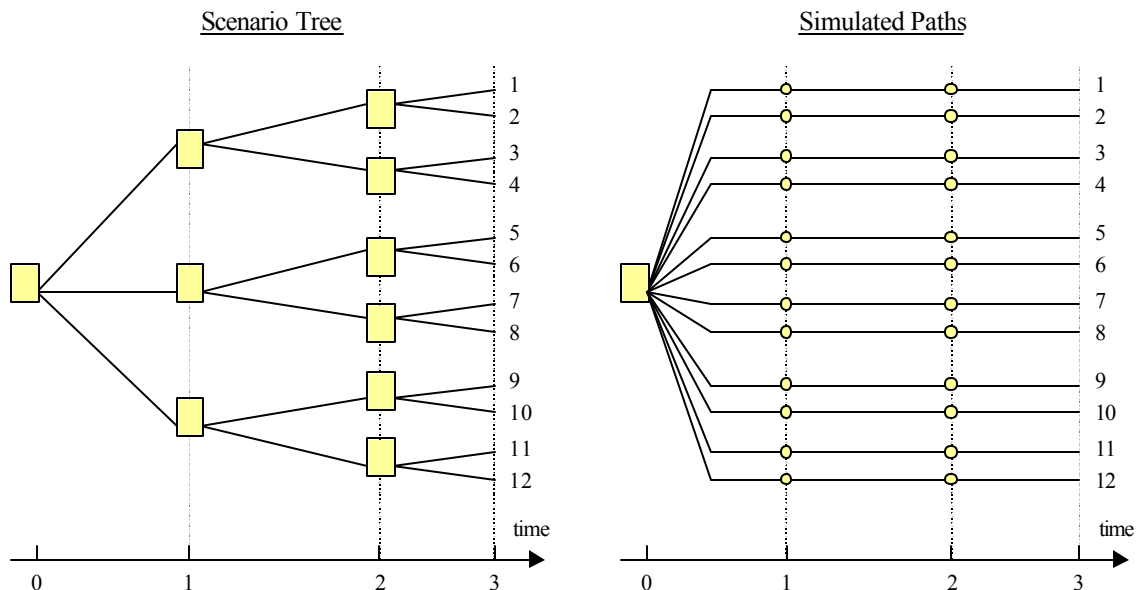### 2.1   Modeling for uncertainties and conditional decisions

Figure 1: Scenario tree and simulated paths

Scenarios of asset returns are typically constructed via a tree structure in the multi-period stochastic programming problem as in the left side of Figure 1. Another description of scenarios is simulated paths. We can sample simulated paths of asset returns on each simulation trial.

An example of simulated paths is shown as in the right side of Figure 1.

Hibiki (2001b) develops the hybrid model in a multi-period optimization framework. Discrete values of asset returns are generated by Monte Carlo simulation to describe the uncertainties more accurately than would the scenario tree, as in the left side of Figure 1. A different decision rule has to be defined in the simulated path approach from the one in the scenario tree approach. The reason is that the model can be solved by anticipating the event in the future if each decision is made on each path. Therefore, the decision rule must be defined to satisfy the non-anticipativity condition [2] in the simulated path framework. Several bundles of simulated paths are made at each time to have a fixed strategy (decision rule) for risky assets [3]. The bundles, or "fixed-decision nodes", are shown as in the left side of Figure 2. Figure 2 represents 12 simulated paths over three periods. It is called a "3-2" branching tree, because it has three bundles at time 1, and more two bundles of paths within each bundle at time 2. The right side of Figure 2 is described as the tree structure, which is called the "extended decision tree", and shows the same structure as the left one. See Hibiki(2003), the bundling procedure for details.
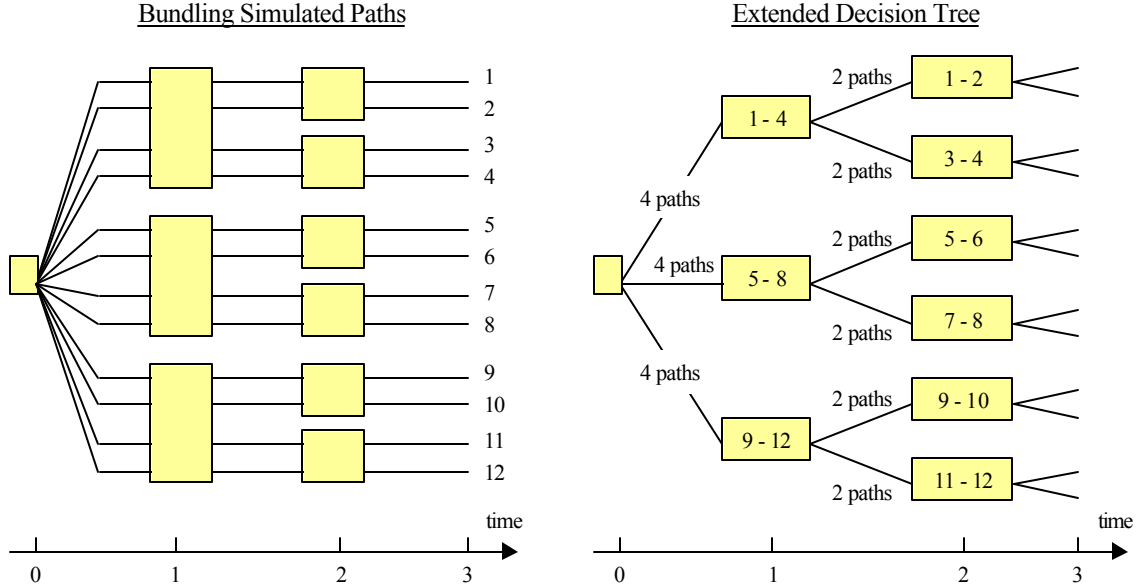

Figure 2: Simulated paths and extended decision tree

2.2   Preparation

We invest in n risky assets and cash. The investment is made at time 0 (present), and time T is the planning horizon.

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[2] The condition which prevents the optimization model from being solved by anticipating the future is called the non-anticipativity condition.

[3] The decisions on cash can be path-dependent, because cash return is based on the interest rate that is risk-free at each time when decision is made.

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2.2.1 Notations

We have three kinds of notations; '(Scenario)' denotes notations for the scenario tree model, '(Hybrid)' denotes notations for the hybrid model, and '(Both)' denotes notations for both models. The notations in this model are as follows.

(1) Sets

S_t : (Scenario) set of states at time t, $(s \in S_t)$,

    : (Hybrid) set of fixed-decision nodes at time t, $(s \in S_t)$.

V_t^s : (Hybrid) set of paths including any fixed-decision node s at time t, $(i \in V_t^s)$.

(2) Parameters

p^s : (Scenario) probability of scenario s at the planning horizon [4] .

I : (Hybrid) number of simulated paths.

ρ_{j0} : (Both) price of risky asset j at time 0, $(j = 1, \ldots, n)$.

ρ_{jt}^s : (Scenario) price of asset j of state s at time t, $(j = 1, \ldots, n;\ t = 1, \ldots, T;\ s \in S_t)$.

$\rho_{jt}^{(i)}$: (Hybrid) price of risky asset j of path i at time t, $(j = 1, \ldots, n;\ t = 1, \ldots, T;\ i = 1, \ldots, I)$.

r_0 : (Both) interest rate in period 1, (the rate at time 0 is used).

$r_{t-1}^{s'}$: (Scenario) interest rate in period $t - 1$, (the rate of state s' at time $t - 1$ is used), $(t = 2, \ldots, T;\ s' \in S_{t-1})$.

$r_{t-1}^{(i)}$: (Hybrid) interest rate in period t (the rate of path i at time $t - 1$ is used), $(t = 2, \ldots, T;\ i = 1, \ldots, I)$.

W_0 : (Both) initial wealth.

W_G : (Both) target terminal wealth.

γ : (Both) risk averse coefficient.

(3) Decision variables

z_{j0} : (Both) investment unit for asset j and time 0, $(j = 1, \ldots, n)$.

z_{jt}^s : (Scenario) investment unit for asset j, time t, and state s,

    : (Hybrid) base investment unit [5] for asset j, time t, and node s
    $(j = 1, \ldots, n;\ t = 1, \ldots, T - 1;\ s \in S_t)$.

v_0 : (Both) cash at time 0

v_t^s : (Scenario) cash of state s at time t, $(t = 1, \ldots, T - 1;\ s \in S_t)$.

$v_t^{(i)}$: (Hybrid) cash of path i at time t, $(t = 1, \ldots, T - 1;\ i = 1, \ldots, I)$.

q^s : (Scenario) shortfall below target terminal wealth of scenario s, $(s \in S_T)$.

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[4]The state s corresponds to the scenario s at the planning horizon.

[5]The base investment unit is defined as the control variable of the investment unit. Details are shown in Section 2.4.1.

$q^{(i)}$: (Hybrid) shortfall below target terminal wealth of path i, $(i = 1, \ldots, I)$.

The variables for risky assets are node-dependent for both models. Cash variables are also node-dependent for a scenario tree model, however they have to be path-dependent for a hybrid model.

2.2.2 Objective function

The objective function to determine the asset mix is the one which maximizes the expected utility. The expected utility is defined using two kinds of measures; the expected terminal wealth $E[W_T]$ and the first-order lower partial moment (LPM_1) of terminal wealth [Bawa and Lindenberg(1977), Harlow(1991)].

$$\text{Expected utility} = E[W_T] - \gamma \cdot LPM_1 \tag{1}$$

The former corresponds to the return measure and the latter corresponds to the risk measure. The lower partial moment is a downside risk measure, and expresses the tail risk of the relevant distribution of wealth below target.

$E[W_T]$ and LPM_1 for both models are calculated as follows [6] ;

$$\text{(Scenario)} \; : \; E[W_T] = \sum_{s \in S_T} p^s W_T^s, \quad LPM_1 = \sum_{s \in S_T} p^s |W_T^s - W_G|_- \tag{2}$$

$$\text{(Hybrid)} \; : \; E[W_T] = \frac{1}{I} \sum_{i=1}^{I} W_T^{(i)}, \quad LPM_1 = \frac{1}{I} \sum_{i=1}^{I} \left| W_T^{(i)} - W_G \right|_- \tag{3}$$

where $|a|_- = \max(-a, 0)$. W_T^s is the terminal wealth of scenario s in the scenario tree model, and $W_T^{(i)}$ is the terminal wealth of path i in the hybrid model.

2.3 The scenario tree model

Generally considered, we can define three kinds of decision variables; investment units, investment values, and investment proportions. Three kinds of formulations with the corresponding decision variables lead equivalent optimal solutions in the scenario tree model. The constraints are non-linear in the case of the investment proportions, however those can be linear in the case of the investment values and investment units. The investment units are used as the decision variables in view of the solution technique.

A typical formulation is follows [7] ;

$$\textbf{Maximize} \quad \sum_{s \in S_T} p^s W_T^s - \gamma \left(\sum_{s \in S_T} p^s q^s \right) \tag{4}$$

$$\textbf{subject to}$$

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[6]The LPM for a continuous distribution of terminal wealth $\tilde{W}_T$ is described as follows:

$$LPM_k \equiv \int_{-\infty}^{W_G} (W_G - \tilde{W}_T)^k f(\tilde{W}_T) d\tilde{W}_T$$

The risk measure corresponds to the first-order LPM for an empirical (discrete) distribution of terminal wealth.

[7]Other constraints such as boundary conditions, policy and legal constraints, and other requirements can be easily added.

$$\sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0 \tag{5}$$

$$(W_1^s =) \sum_{j=1}^{n} \rho_{j1}^s z_{j1}^s + v_1^s = \sum_{j=1}^{n} \rho_{j1}^s z_{j0} + (1 + r_0) v_0 , \ (s \in S_1) \tag{6}$$

$$(W_t^s =) \sum_{j=1}^{n} \rho_{jt}^s z_{jt}^s + v_t^s = \sum_{j=1}^{n} \rho_{jt}^s z_{j,t-1}^{s'} + (1 + r_{t-1}^{s'}) v_{t-1}^{s'}, \ (t = 2, \ldots, T-1; \ s \in S_t) \tag{7}$$

$$W_T^s = \sum_{j=1}^{n} \rho_{jT}^s z_{j,T-1}^{s'} + (1 + r_{T-1}^{s'}) v_{T-1}^{s'} , \ (s \in S_T) \tag{8}$$

$$W_T^s + q^s \geq W_G , \ (s \in S_T) \tag{9}$$

$$z_{j0} \geq 0, \ (j = 1, \ldots, n); \quad z_{jt}^s \geq 0, \ (j = 1, \ldots, n; \ t = 1, \ldots, T-1; \ s \in S_t) \tag{10}$$

$$v_0 \geq 0; \quad v_t^s \geq 0, \ (t = 1, \ldots, T-1; \ s \in S_t) \tag{11}$$

$$q^s \geq 0, \ (s \in S_T) \tag{12}$$

Constraint (5) is a budget constraint at time 0. Constraints (6) and (7) are cash flow constraints at time t, the values of both sides of which show the wealth of state s at time t. We denote s' in Constraint (7) to be the state at time $t-1$ connected with the state at time t. Constraint (8) shows the terminal wealth. We can minimize LPM_1 in Equation (2) by minimizing the second term of the objective function under the Constraint (9). Constraints (10) - (12) are non-negative constraints.

2.4 The hybrid model

2.4.1 Investment strategies with investment unit functions

We can select a fixed strategy for risky assets, such as fixed-proportion strategy, fixed-value strategy, fixed-unit strategy, and so on. Even if we select investment units as decision variables, we do not have to fix investment units at each node. If we define the function of the decision rule associated with the investment units, we can invest the different units on each path through a node. This is called the "investment unit function". Moreover, we can describe other strategies such as fixed-proportion strategy and fixed-value strategy by using this function. The investment unit function is defined to show various investment strategies, as follows.

$$h^{(i)}(z_{jt}^s) = a_{jt}^{(i)} z_{jt}^s \tag{13}$$

where $a_{jt}^{(i)}$ is the investment unit parameter that must be independent on the rate of returns of path i after time t to keep non-anticipativity condition. We consider three kinds of investment strategies with investment unit functions.

(1) Fixed-unit strategy : $h^{(i)}(z_{jt}^s) = z_{jt}^s$

All risky assets have the same investment units on any path at each node, respectively. However, cash is different in each path.

(2) Fixed-value strategy : $h^{(i)}(z_{jt}^s) = \left(\dfrac{\rho_{j0}}{\rho_{jt}^{(i)}} \right) z_{jt}^s$

All risky assets have the same investment values on any path at each node, respectively. However, cash is different in each path [8] .

(3) Fixed-proportion strategy : $h^{(i)}(z_{jt}^s) = \left(\dfrac{W_t^{(i)}}{\rho_{jt}^{(i)}} \right) z_{jt}^s$

All risky assets and cash have the same investment proportions on any path at each node, respectively.

Constraints are linear in the fixed-unit strategy and the fixed-value strategy. But constraints are non-convex in the fixed-proportion strategy because $W_t^{(i)}$ is a function of decision variables. The iterative method is proposed to solve the non-convex program approximately in Section 2.5.

2.4.2   Formulation

We show a typical formulation, which structure is the same as that of the scenario tree model [9];

$$\textbf{Maximize} \quad \frac{1}{I} \sum_{i=1}^{I} W_T^{(i)} - \gamma \left(\frac{1}{I} \sum_{i=1}^{I} q^{(i)} \right) \tag{14}$$

subject to

$$\sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0 \tag{15}$$

$$\sum_{j=1}^{n} \rho_{j1}^{(i)} z_{j0} + (1 + r_0) v_0 = \sum_{j=1}^{n} \rho_{j1}^{(i)} h^{(i)}(z_{j1}^s) + v_1^{(i)}, \quad (s \in S_1;\ i \in V_1^s) \tag{16}$$

$$\sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{j,t-1}^{s'}) + \left(1 + r_{t-1}^{(i)} \right) v_{t-1}^{(i)} = \sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{jt}^s) + v_t^{(i)},$$
$$(t = 2, \ldots, T-1;\ s \in S_t;\ i \in V_t^s) \tag{17}$$

$$W_T^{(i)} = \sum_{j=1}^{n} \rho_{jT}^{(i)} h^{(i)}(z_{j,T-1}^{s'}) + \left(1 + r_{T-1}^{(i)} \right) v_{T-1}^{(i)}, \quad (s' \in S_{T-1};\ i \in V_{T-1}^{s'}) \tag{18}$$

$$W_T^{(i)} + q^{(i)} \geq W_G, \quad (i = 1, \ldots, I) \tag{19}$$

$$z_{j0} \geq 0,\ (j = 1, \ldots, n); \quad z_{jt}^s \geq 0,\ (j = 1, \ldots, n;\ t = 1, \ldots, T-1;\ s \in S_t) \tag{20}$$

$$v_0 \geq 0; \quad v_t^{(i)} \geq 0,\ (t = 1, \ldots, T-1;\ i = 1, \ldots, I) \tag{21}$$

$$q^{(i)} \geq 0, \quad (i = 1, \ldots, I) \tag{22}$$

If we select the strategy which has a linear investment unit function, we can formulate as a linear programming problem, and solve a large-scale problem easily in practical use.

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[8]This strategy is a kind of contrarian investment strategies, because an investment unit tends to be decreased when price goes up, and tends to be increased when price goes down.

[9]Constraint (15) is a budget constraint at time 0. Constraints (16) and (17) are cash flow constraints at time t. Constraint (18) shows the terminal wealth. We can minimize LPM_1 in Equation (3) by minimizing the second term of the objective function under the Constraint (19). Constraints (20) - (22) are non-negative constraints.

2.5 Iterative algorithm to solve the hybrid model with the fixed-proportion strategy

According to our previous numerical tests for the hybrid model, the efficient frontier of the fixed-proportion strategy dominates the efficient frontier of the fixed-unit strategy in our examples [see Hibiki(2003) for details]. Therefore, we have to solve the hybrid model with the fixed-proportion strategy [10] to compare it with the scenario tree model.

Suppose that we can derive the optimal solutions of the hybrid model with the fixed-proportion strategy, and calculate the wealth of path i at time t, $W_t^{(i)*}$. If we set up $h^{(i)}(z_{jt}^s) = \left(\frac{W_t^{(i)*}}{\rho_{jt}^{(i)}} \right) z_{jt}^s$ as the investment unit function, and solve the problem, the same solutions are supposed to be obtained. We develop the iterative algorithm using this feature to solve the hybrid model with the fixed-proportion strategy approximately. Algorithm is as follows;

Step 1: We solve the problem with the fixed-unit strategy and calculate wealth of path i at time t, $W_{t(0)}^{(i)*}$. Denote Obj_0 to be the objective function value, and set $k = 1$.

Step 2: We set up $h^{(i)}(z_{jt}^s) = \left(\frac{W_{t(k-1)}^{(i)*}}{\rho_{jt}^{(i)}} \right) z_{jt}^s$ as the investment unit function at the k-th iteration, and solve the problem. We calculate wealth of path i at time t, $W_{t(k)}^{(i)*}$, and objective function value Obj_k.

Step 3: Stop if the value $Obj_k - Obj_{k-1}$ is lower than the tolerance. Otherwise, set $k \leftarrow k+1$, and return to Step 2.

This algorithm does not guarantee to derive the global optimal solutions for the fixed-proportion strategy, but optimal solutions are stably derived whatever initial point is set, because we solve the linear programming problem iteratively.

We evaluate this solution algorithm from two points of view; the objective function value and the optimal investment proportions. Table 1 shows the improvement rate of the objective function for 15 kinds of risk-averse coefficients. These are the normalized values of the objective functions of the first iteration to the fifth iteration so that objective function value of the fifth iteration(Obj_5) minus objective function value of the fixed-unit strategy(Obj_0) is equal to 100%. Objective function values is almost improved up to the fifth iteration. And more than 99% is improved up to the second iteration.

Table 1: Improvement rate of the objective function

γ	10	5	4	3	2	1.5	1	0.8	0.6	0.5	0.4	0.3	0.2	0.1	max	Ave.
$k=1$	99.2%	96.1%	95.1%	95.3%	95.1%	95.6%	95.1%	95.0%	94.3%	94.4%	94.4%	94.3%	100.0%	91.2%	85.9%	94.7%
$k=2$	0.6%	3.8%	4.6%	4.6%	4.7%	4.2%	4.8%	4.9%	5.5%	5.4%	5.5%	5.6%	0.0%	8.3%	13.3%	5.1%
$k=3$	0.2%	0.1%	0.2%	0.1%	0.2%	0.2%	0.1%	0.2%	0.1%	0.1%	0.1%	0.1%	0.0%	0.4%	0.8%	0.2%
$k=4$	0.1%	-0.1%	0.0%	0.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
$k=5$	0.0%	0.0%	0.0%	-0.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
up to 2	99.8%	100.0%	99.7%	99.9%	99.8%	99.8%	99.8%	99.8%	99.8%	99.9%	99.9%	99.9%	100.0%	99.5%	99.2%	99.8%

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[10] We tried to solve this problem using nonlinear programming tools, but we could not derive optimal solutions stably because of non-convex constraints.

Table 2 shows the average of standard deviation of investment ratios on the paths through each node at time t. #n denotes the n-th iteration. If this value is equal to 0, this shows that investment ratios at each node are the same value. This means that the fixed-proportion strategy can be almost implemented. The smaller the value, the more approximate the investment ratios on the paths. All of values are lower than 0.1% at the second iteration. Because of these results and saving computation time, we execute two iterations to solve the model with the fixed-proportion strategy in this paper.

Table 2: Average of standard deviation of investment ratios on the paths through each node

		min #0	#1	#2	#3		γ=10 #0	#1	#2	#3		γ=5 #0	#1	#2	#3		γ=4 #0	#1	#2	#3
cash	1	0.75%	0.03%	0.00%	0.02%	cash 1	1.04%	0.04%	0.00%	0.00%	cash 1	1.50%	0.07%	0.01%	0.01%	cash 1	1.54%	0.09%	0.01%	0.01%
	2	0.43%	0.05%	0.01%	0.00%	2	0.54%	0.04%	0.02%	0.01%	2	0.74%	0.03%	0.01%	0.02%	2	0.84%	0.04%	0.01%	0.01%
	3	0.76%	0.03%	0.02%	0.01%	3	1.06%	0.04%	0.02%	0.01%	3	1.41%	0.06%	0.01%	0.01%	3	1.50%	0.07%	0.02%	0.01%
stock	1	0.14%	0.00%	0.00%	—	stock 1	0.90%	0.00%	0.00%	—	stock 1	2.07%	0.00%	0.00%	—	stock 1	1.54%	0.00%	0.00%	—
	2	0.26%	0.00%	0.00%	—	2	0.28%	0.00%	0.00%	—	2	0.49%	0.01%	0.01%	0.01%	2	0.68%	0.01%	0.00%	0.02%
	3	0.45%	0.00%	0.00%	0.00%	3	0.77%	0.01%	0.01%	0.00%	3	1.11%	0.03%	0.01%	0.01%	3	1.16%	0.03%	0.01%	0.02%
bond	1	0.13%	0.00%	0.02%	—	bond 1	0.26%	0.02%	0.01%	0.00%	bond 1	0.30%	0.01%	0.02%	0.01%	bond 1	0.34%	0.02%	0.03%	0.03%
	2	0.35%	0.01%	0.02%	0.00%	2	0.43%	0.01%	0.01%	0.02%	2	0.54%	0.03%	0.02%	0.03%	2	0.62%	0.03%	0.03%	0.01%
	3	0.57%	0.02%	0.02%	0.01%	3	0.68%	0.03%	0.03%	0.01%	3	0.87%	0.04%	0.02%	0.02%	3	0.93%	0.04%	0.03%	0.02%
CB	1	0.34%	0.02%	0.00%	0.00%	CB 1	0.48%	0.01%	0.01%	0.00%	CB 1	0.66%	0.00%	0.02%	—	CB 1	0.77%	0.03%	0.03%	0.00%
	2	0.45%	0.02%	0.01%	0.00%	2	0.73%	0.02%	0.02%	0.01%	2	0.89%	0.02%	0.01%	0.00%	2	0.89%	0.02%	0.01%	0.02%
	3	0.97%	0.02%	0.02%	0.01%	3	1.37%	0.03%	0.02%	0.03%	3	1.45%	0.05%	0.02%	0.01%	3	1.46%	0.05%	0.03%	0.03%

		γ=3 #0	#1	#2	#3		γ=2 #0	#1	#2	#3		γ=1.5 #0	#1	#2	#3		γ=1 #0	#1	#2	#3
cash	1	1.58%	0.09%	0.01%	0.00%	cash 1	1.62%	0.14%	0.02%	0.00%	cash 1	1.63%	0.12%	0.01%	0.02%	cash 1	1.70%	0.13%	0.01%	0.00%
	2	1.00%	0.04%	0.01%	0.01%	2	1.27%	0.04%	0.01%	0.01%	2	1.44%	0.04%	0.01%	0.00%	2	1.81%	0.06%	0.01%	0.00%
	3	1.69%	0.09%	0.01%	0.00%	3	1.99%	0.12%	0.01%	0.00%	3	2.23%	0.14%	0.01%	0.00%	3	2.55%	0.17%	0.01%	0.00%
stock	1	2.03%	0.00%	0.00%	—	stock 1	2.04%	0.00%	0.01%	—	stock 1	2.03%	0.01%	0.00%	0.00%	stock 1	0.57%	0.00%	0.02%	—
	2	0.88%	0.02%	0.01%	0.00%	2	1.33%	0.04%	0.02%	0.04%	2	1.55%	0.04%	0.02%	0.00%	2	1.95%	0.05%	0.02%	0.01%
	3	1.34%	0.05%	0.01%	0.01%	3	1.78%	0.11%	0.01%	0.02%	3	1.94%	0.12%	0.02%	0.02%	3	2.40%	0.18%	0.02%	0.01%
bond	1	0.48%	0.05%	0.04%	0.03%	bond 1	0.57%	0.04%	0.00%	0.00%	bond 1	0.70%	0.04%	0.04%	0.02%	bond 1	0.88%	0.02%	0.05%	0.01%
	2	0.62%	0.03%	0.02%	0.02%	2	0.75%	0.04%	0.02%	0.03%	2	0.92%	0.04%	0.01%	0.03%	2	1.08%	0.03%	0.02%	0.03%
	3	1.10%	0.05%	0.03%	0.02%	3	1.30%	0.05%	0.03%	0.03%	3	1.51%	0.08%	0.04%	0.03%	3	1.84%	0.09%	0.03%	0.03%
CB	1	0.86%	0.01%	0.03%	0.03%	CB 1	0.94%	0.00%	0.00%	—	CB 1	0.97%	0.05%	0.00%	0.00%	CB 1	0.98%	0.02%	0.00%	0.05%
	2	1.07%	0.03%	0.01%	0.02%	2	1.25%	0.03%	0.02%	0.01%	2	1.30%	0.03%	0.03%	0.01%	2	1.41%	0.05%	0.02%	0.02%
	3	1.59%	0.08%	0.02%	0.03%	3	1.68%	0.08%	0.02%	0.03%	3	1.79%	0.09%	0.03%	0.04%	3	1.83%	0.10%	0.02%	0.02%

		γ=0.8 #0	#1	#2	#3		γ=0.6 #0	#1	#2	#3		γ=0.5 #0	#1	#2	#3		γ=0.4 #0	#1	#2	#3
cash	1	1.76%	0.12%	0.01%	0.00%	cash 1	1.69%	0.15%	—	0.00%	cash 1	1.64%	0.14%	—	0.00%	cash 1	1.53%	0.14%	—	0.00%
	2	1.99%	0.06%	0.00%	0.00%	2	2.12%	0.08%	0.00%	0.00%	2	2.21%	0.07%	0.00%	0.00%	2	2.28%	0.06%	0.00%	0.00%
	3	2.70%	0.19%	0.01%	0.00%	3	2.83%	0.23%	0.01%	0.00%	3	2.93%	0.23%	0.01%	0.00%	3	3.11%	0.22%	0.01%	0.00%
stock	1	0.59%	0.01%	0.02%	0.02%	stock 1	0.68%	0.00%	0.00%	—	stock 1	0.73%	0.01%	0.02%	0.00%	stock 1	0.80%	0.00%	0.00%	—
	2	2.10%	0.05%	0.01%	0.03%	2	2.02%	0.04%	0.00%	0.02%	2	2.16%	0.04%	0.01%	0.02%	2	2.05%	0.03%	0.02%	0.04%
	3	2.56%	0.20%	0.02%	0.03%	3	2.65%	0.24%	0.03%	0.03%	3	2.71%	0.24%	0.02%	0.03%	3	2.53%	0.20%	0.02%	0.03%
bond	1	1.19%	0.04%	0.03%	0.04%	bond 1	1.32%	0.04%	0.04%	0.00%	bond 1	1.42%	0.02%	0.04%	0.07%	bond 1	1.52%	0.04%	0.03%	0.01%
	2	1.43%	0.03%	0.03%	0.04%	2	1.59%	0.05%	0.02%	0.01%	2	1.64%	0.04%	0.03%	0.03%	2	2.17%	0.06%	0.03%	0.04%
	3	2.07%	0.10%	0.04%	0.02%	3	2.29%	0.11%	0.03%	0.03%	3	2.59%	0.13%	0.03%	0.03%	3	2.94%	0.13%	0.04%	0.03%
CB	1	0.99%	0.03%	0.03%	0.09%	CB 1	0.86%	0.04%	0.03%	0.00%	CB 1	0.72%	0.02%	0.04%	0.10%	CB 1	0.53%	0.07%	0.03%	0.01%
	2	1.54%	0.02%	0.02%	0.02%	2	1.65%	0.07%	0.05%	0.01%	2	1.56%	0.03%	0.01%	0.04%	2	1.34%	0.06%	0.02%	0.05%
	3	1.95%	0.13%	0.03%	0.03%	3	2.04%	0.16%	0.02%	0.04%	3	1.86%	0.14%	0.04%	0.03%	3	1.76%	0.14%	0.02%	0.02%

		γ=0.3 #0	#1	#2	#3		γ=0.2 #0	#1	#2	#3		γ=0.1 #0	#1	#2	#3		max #0	#1	#2	#3
cash	1	1.44%	0.14%	0.01%	0.00%	cash 1	1.45%	0.16%	—	0.00%	cash 1	1.45%	0.16%	—	0.00%	cash 1	1.84%	0.81%	—	0.00%
	2	2.47%	0.06%	0.00%	0.00%	2	2.80%	0.07%	0.00%	0.00%	2	2.80%	0.07%	0.00%	0.00%	2	3.94%	0.46%	0.00%	0.00%
	3	3.37%	0.23%	0.00%	0.00%	3	3.54%	0.28%	0.01%	0.00%	3	3.54%	0.28%	0.01%	0.00%	3	4.51%	0.63%	0.05%	0.00%
stock	1	0.90%	0.01%	0.01%	0.00%	stock 1	1.05%	0.00%	0.04%	—	stock 1	1.05%	0.00%	0.04%	—	stock 1	0.00%	0.00%	0.00%	—
	2	2.27%	0.02%	0.01%	0.04%	2	2.70%	0.04%	0.03%	0.02%	2	2.70%	0.04%	0.03%	0.02%	2	0.61%	1.32%	0.00%	0.00%
	3	2.89%	0.21%	0.02%	0.04%	3	2.95%	0.20%	0.04%	0.04%	3	2.95%	0.20%	0.04%	0.04%	3	1.17%	1.08%	0.07%	0.00%
bond	1	1.50%	0.03%	0.04%	0.01%	bond 1	2.06%	0.02%	0.01%	0.00%	bond 1	2.06%	0.02%	0.01%	0.00%	bond 1	4.46%	0.00%	0.00%	—
	2	2.29%	0.06%	0.04%	0.02%	2	2.95%	0.09%	0.03%	0.05%	2	2.95%	0.09%	0.03%	0.05%	2	5.08%	0.09%	0.00%	0.00%
	3	3.35%	0.14%	0.03%	0.03%	3	3.58%	0.21%	0.03%	0.03%	3	3.58%	0.21%	0.03%	0.03%	3	5.64%	0.34%	0.00%	0.00%
CB	1	0.31%	0.04%	0.03%	0.00%	CB 1	0.55%	0.04%	0.04%	0.05%	CB 1	0.55%	0.04%	0.04%	0.05%	CB 1	1.74%	0.00%	0.00%	—
	2	1.10%	0.06%	0.05%	0.02%	2	1.22%	0.06%	0.04%	0.04%	2	1.22%	0.06%	0.04%	0.04%	2	3.57%	0.05%	0.00%	0.00%
	3	1.70%	0.14%	0.03%	0.03%	3	1.91%	0.21%	0.03%	0.02%	3	1.91%	0.21%	0.03%	0.02%	3	3.34%	0.33%	0.04%	0.00%

2.6 Comparison of the models

We explain the similarity and the difference between the scenario tree model and the hybrid model in Figure 3. Suppose six scenarios over two periods and three nodes at time 1. $\mu_{jt}^{(i)}$ denotes a rate of return of asset j, time t, and path i, and $w_{jt}^{(i)}$ denotes the decision on each path. There are six states at both time 1 and time 2 in the hybrid model. There are three states at time 1

in the scenario tree model. If six states are assumed in the hybrid model, two state values have to be the same value. On the other hand, the conditional decisions are made similarly in both models. For example, $w_{jt}^{(1)}$ has to be equal to $w_{jt}^{(2)}$ to keep the non-anticipativity condition. If the number of states in the scenario tree model is equal to the number of fixed decision nodes in the hybrid model, both models can make the same kind of conditional decisions despite of the different description of uncertainties.

$$\mu_{j1}^{(1)} = \mu_{j1}^{(2)}, \ \mu_{j1}^{(3)} = \mu_{j1}^{(4)}, \ \mu_{j1}^{(5)} = \mu_{j1}^{(6)} \qquad \mu_{j1}^{(1)} \neq \mu_{j1}^{(2)}, \ \mu_{j1}^{(3)} \neq \mu_{j1}^{(4)}, \ \mu_{j1}^{(5)} \neq \mu_{j1}^{(6)}$$
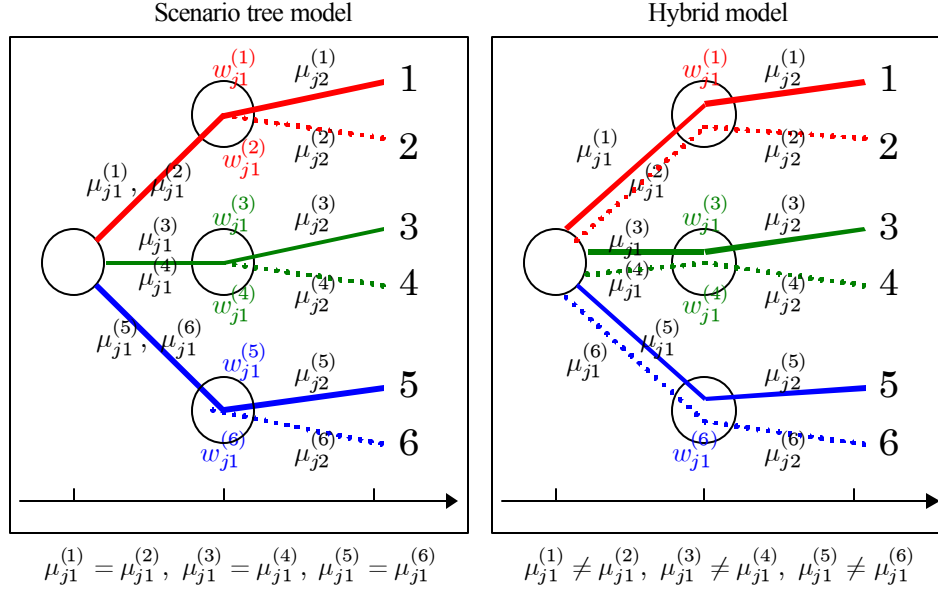
Figure 3: Comparison between the scenario tree model and the hybrid model

3 Scenario Generation

3.1 Generating simulated paths for the hybrid model

In general, scenarios associated with asset returns are generated according to the stochastic differential equations or time series models. Mulvey and Thorlacius(1998) use Towers Perrin's scenario generation system, "CAP: Link" to solve a multi-period stochastic programming problem for pension funds. A scenario system is based on a cascading set of stochastic differential equations. The Russel-Yasuda model(See Carino *et al.*, 1998a,1998b and 1998c) used for the ALM of casualty insurance company, generates scenarios whose returns are created from a factor model that incorporates dependence between periods.

Two kinds of models need different kinds of scenario structures. However, it is difficult to compare the results based on the different scenario structures. Then, it is necessary to consider how to generate scenarios based on the same possible situation and how to compare the results. Consider two kinds of choices as follows;

- the 'path to tree' procedure : First, we generate simulated paths, and secondly, we construct the scenario tree based on the simulated paths.

- the 'tree to path' procedure : First, we construct a scenario tree, and secondly, we generate the simulated paths based on the scenario tree.

In this paper, we choose the 'path to tree' procedure, because of the following reasons.

(1) The sample path structure the hybrid model needs can be generated by Monte Carlo simulation as in the right side of Figure 1. If a scenario generation model is selected, we can generate such sample paths easily by a standard procedure of Monte Carlo simulation technique.

(2) It is difficult to construct the appropriate scenario tree from the stochastic differential equations or time-series models. This is because we need to describe asset returns appropriately by a moderate number of scenarios [11] .

(3) The underlying scenarios should be described by the simulated paths because simulated paths are superior to a scenario tree when describing the uncertainties.

(4) The scenario tree can be generated using the classifying mothod described in Section 3.2. (See Section 3.3 for a detail discussion.)

It is important which model is selected because the optimal solutions change according to the model. However, the main aim of this paper is to compare the multi-period optimization models and to clarify the difference between two models. Then we use the following simple procedure with the statistics associated with asset returns(expected rate of return, standard deviation and correlation matrix of rate of return) to generate scenarios of rates of returns of n risky assets and call rate.

The rate of return $\mu_{jt}^{(i)}$ is generated as follows, where asset 0 $(j = 0)$ corresponds to call rate.

① The rate of return of asset j in period t is normally distributed with mean $\overline{\mu}_{jt}$ and standard deviation σ_{jt}, and it is generated by:

$$\mu_{jt}^{(i)} = \overline{\mu}_{jt} + \sigma_{jt}\varepsilon_{jt}^{(i)},$$

where $\varepsilon_{jt}^{(i)}$ is a random sample from a multi-variate standardized normal distribution.

② The random variable ε_{jt} $(j = 0, \dots, n;\ t = 1, \dots, T)$ follows that

$$\varepsilon_{jt} \ \sim \ N(\mathbf{o}, \Sigma),$$

where Σ is $(n + 1)T \times (n + 1)T$ correlation matrix.

$\mu_{0t}^{(i)}$ is the change rate of call rate. The call rate $r_t^{(i)}$ is calculated by:

$$r_1^{(i)} \ = \ r_0 \times \left(1 + \mu_{01}^{(i)}\right),$$

$$r_t^{(i)} \ = \ r_{t-1}^{(i)} \times \left(1 + \mu_{0t}^{(i)}\right), \ (t = 2, \dots, T - 1).$$

Random samples are generated from two kinds of summary statistics for numerical tests. One is the statistics calculated by historical data. Table 3 shows the summary statistics calculated

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[11] The detail implementation method has not been expressed in those papers(Mulvey and Thorlacius, 1998; Carino *et al.*, 1998a,1998b and 1998c). If a set of scenarios is constructed by a tree structure to a sufficient degree, the problem size may grow exponentially.

by the available market data; Nikko stock performance index (TSE 1), Nikko bond performance index, Nikko CB performance index, and call rate.

Table 3: Summary statistics

		cash				stock				bond				CB		
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Exp. Value	−0.087	−0.081	−0.089	−0.103	0.848	0.867	0.843	0.858	0.625	0.623	0.645	0.683	0.786	0.780	0.786	0.806
St. Dev.	0.780	0.784	0.778	0.759	5.571	5.582	5.595	5.591	1.372	1.372	1.353	1.233	3.543	3.541	3.538	3.525

Correlation		1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
cash	1	1.000	−0.091	0.073	0.226	−0.101	0.000	−0.032	−0.042	−0.238	0.008	0.090	0.038	−0.146	−0.044	−0.052	−0.036
	2	−0.091	1.000	−0.092	0.074	0.045	−0.094	−0.007	−0.032	−0.183	−0.237	0.011	0.093	−0.012	−0.144	−0.047	−0.053
	3	0.073	−0.092	1.000	−0.123	0.016	0.042	−0.091	−0.002	−0.166	−0.188	−0.221	0.068	−0.062	−0.017	−0.138	−0.037
	4	0.226	0.074	−0.123	1.000	−0.015	0.012	0.048	−0.085	−0.055	−0.176	−0.156	−0.145	−0.029	−0.072	−0.005	−0.122
stock	1	−0.101	0.045	0.016	−0.015	1.000	0.022	−0.031	0.030	0.145	−0.173	−0.096	−0.065	0.761	0.042	−0.045	−0.056
	2	0.000	−0.094	0.042	0.012	0.022	1.000	0.018	−0.030	0.085	0.144	−0.170	−0.101	0.019	0.760	0.041	−0.045
	3	−0.032	−0.007	−0.091	0.048	−0.031	0.018	1.000	0.018	0.077	0.085	0.141	−0.189	0.011	0.019	0.760	0.041
	4	−0.042	−0.032	−0.002	−0.085	0.030	−0.030	0.018	1.000	0.130	0.078	0.080	0.137	−0.022	0.013	0.017	0.760
bond	1	−0.238	−0.183	−0.166	−0.055	0.145	0.085	0.077	0.130	1.000	0.130	−0.108	−0.118	0.327	0.202	0.065	0.136
	2	0.008	−0.237	−0.188	−0.176	−0.173	0.144	0.085	0.078	0.130	1.000	0.137	−0.106	−0.114	0.327	0.204	0.068
	3	0.090	0.011	−0.221	−0.156	−0.096	−0.170	0.141	0.080	−0.108	0.137	1.000	0.072	−0.180	−0.109	0.321	0.192
	4	0.038	0.093	0.068	−0.145	−0.065	−0.101	−0.189	0.137	−0.118	−0.106	0.072	1.000	−0.117	−0.182	−0.142	0.315
CB	1	−0.146	−0.012	−0.062	−0.029	0.761	0.019	0.011	−0.022	0.327	−0.114	−0.180	−0.117	1.000	0.092	−0.068	−0.032
	2	−0.044	−0.144	−0.017	−0.072	0.042	0.760	0.019	0.013	0.202	0.327	−0.109	−0.182	0.092	1.000	0.093	−0.066
	3	−0.052	−0.047	−0.138	−0.005	−0.045	0.041	0.760	0.017	0.065	0.204	0.321	−0.142	−0.068	0.093	1.000	0.090
	4	−0.036	−0.053	−0.037	−0.122	−0.056	−0.045	0.041	0.760	0.136	0.068	0.192	0.315	−0.032	−0.066	0.090	1.000

The other is the virtual statistics considering serial correlations between different two periods. It is one of the important concerns in the multi-period model to take the serial correlation of the asset price into consideration. We denotes c to be the parameter associated with the serial correlation. Eleven cases of different parameters are tested to examine the effect of serial correlation as in Table 4.

The parameter c is also the autocorrelation of each asset itself between period t and period $t+1$ ($t = 1, 2, 3$). Serial correlation coefficients between period t and period $t+1$ are c times the correlation coefficients as in Table 5. Serial correlation coefficients between period t and period $t+k$ are 0.3 times the serial correlation coefficients between period t and period $t+k-1$, ($k = 2, 3$). Expected value and standard deviation of rate of return are the same values in each period as in Table 6.

Table 4: Eleven kinds of correlation parameters

Case	cm5	cm4	cm3	cm2	cm1	
Parameter	$c = -0.5$	$c = -0.4$	$c = -0.3$	$c = -0.2$	$c = -0.1$	
Case	cp0	cp1	cp2	cp3	cp4	cp5
Parameter	$c = 0.0$	$c = 0.1$	$c = 0.2$	$c = 0.3$	$c = 0.4$	$c = 0.5$

Table 5: Correlation matrix

Correlation

	cash(t)	stock(t)	bond(t)	CB(t)
cash(t)	1	−0.10	−0.25	−0.15
stock(t)	−0.10	1	0.15	0.75
bond(t)	−0.25	0.15	1	0.30
CB(t)	−0.15	0.75	0.30	1

S.C.(± 1) : correlation$\times c$

	cash($t+1$)	stock($t+1$)	bond($t+1$)	CB($t+1$)
cash(t)	c	$-0.10c$	$-0.25c$	$-0.15c$
stock(t)	$-0.10c$	c	$0.15c$	$0.75c$
bond(t)	$-0.25c$	$0.15c$	c	$0.30c$
CB(t)	$-0.15c$	$0.75c$	$0.30c$	c

S.C.(± 2) : S.C.(± 1)$\times 0.3$

	cash($t+2$)	stock($t+2$)	bond($t+2$)	CB($t+2$)
cash(t)	$0.300c$	$-0.030c$	$-0.075c$	$-0.045c$
stock(t)	$-0.030c$	$0.300c$	$0.045c$	$0.225c$
bond(t)	$-0.075c$	$0.045c$	$0.300c$	$0.090c$
CB(t)	$-0.045c$	$0.225c$	$0.090c$	$0.300c$

S.C.(± 3) : S.C.(± 2)$\times 0.3$

	cash($t+3$)	stock($t+3$)	bond($t+3$)	CB($t+3$)
cash(t)	$0.0900c$	$-0.0009c$	$-0.0225c$	$-0.0135c$
stock(t)	$-0.0090c$	$0.0900c$	$0.0135c$	$0.0675c$
bond(t)	$-0.0225c$	$0.0135c$	$0.0900c$	$0.0270c$
CB(t)	$-0.0135c$	$0.0675c$	$0.0270c$	$0.0900c$

S.C. : serial correlation(autocorrelation, cross-autocorrelation)

Table 6: Expected rate of return and standard deviation of rate of return

Rate of return	cash	stock	bond	CB
Exp. Value	0.0%	0.85%	0.6%	0.75%
St. Dev	0.4%	5.5%	1.4%	3.5%

3.2 Procedure of generating extended decision tree

We need to classify and bundle the simulated paths to make conditional decisions in the hybrid model. Hibiki(2003) showed two kinds of classifying methods as follows.

(1) Sequential clustering method (SQC method)

This method is applied to the data set of simulated paths over the planning period by using the well-known hierarchical clustering method in each period sequentially. Generated clusters represent the fixed-decision nodes. The method is implemented based on similarities calculated by distances between sampled return vectors.

(2) Portfolio based clustering method (PBC method)

This method is applied to the wealth of path i at time t which is calculated by any portfolio over the planning period. We can use any portfolio, such as an equally weighted portfolio, an optimal portfolio derived by solving the simulated path model , and so on. But it is dependent on which portfolios are appropriate to the model. Therefore, we need to compare some portfolios to solve the model.

Because Hibiki(2003) showed the PBC method with an optimal portfolio for the simulated path model (S-PBC method) is the best method among these methods, the S-PBC method is used

in this paper. This method is applied not only to bundle the simulated paths but also to make a scenario tree.

3.3 Generating a scenario tree from simulated paths to compare the scenario tree model with the hybrid model

We should not compare the efficient frontiers derived from two kinds of models, which use different scenario structures. We generate a scenario tree based on the 'path to tree' procedure as mentioned before. The following steps are proposed to compare two kinds of models. .

Step 1: We generate simulated paths and bundle them using the S-PBC method. We solve the hybrid model with the fixed-proportion strategy for several risk-averse coefficients, and derive optimal investment ratios. We calculate several expected terminal wealth and risk to describe the efficient frontier. This step is the standard procedure for the hybrid model.

Step 2: We generate a scenario tree from the simulated paths used when solving the hybrid model in Step 1, and calculate prices on the scenario tree.

Step 3: We solve the scenario tree model for several risk-averse coefficients, and derive optimal investment units. We calculate optimal investment ratios from optimal investment units.

Step 4: We apply the optimal investment ratios derived from the scenario tree model to the hybrid model, and calculate several expected terminal wealth and risk to describe the relationship curve between them.

Step 5: We compare the curve of the scenario tree model with the efficient frontier of the hybrid model derived in Step 1.

We explain the details from Step 2 to Step 4.

Step 2: Procedure of generating a scenario tree from simulated paths

① Calculation of the average value

The average rates of returns are calculated from period 1 to period $T-1$ because $\mu_{jT}^{(i)}$ can be used as the rate of return in the period T. We compute $\bar{\mu}_{jt}^{s}$, the average rate of return at node s.

$$\bar{\mu}_{jt}^{s} = \frac{1}{|V_t^s|} \sum_{i \in V_t^s} \mu_{jt}^{(i)}, \ (j = 1, \ldots, n; \ t = 1, \ldots, T-1; \ s \in S_t)$$

where $\mu_{jt}^{(i)}$ is the rate of return for asset j, period t, and path i. Set of paths V_t^s generated in Step 1 must be used. $|V_t^s|$ denotes the number of paths in the set.

② The moment matching

$\bar{\mu}_{jt}^{s}$ is adjusted so that the expected value and standard deviation of μ_{jt}^{s} calculated are equivalent to those of $\mu_{jt}^{(i)}$.

$$\mu_{jt}^s = \left(\frac{\overline{\mu}_{jt}^s - \overline{\mu}_{jt}}{\sigma_{jt}^S} \right) \times \sigma_{jt} + \overline{\mu}_{jt}, \ (j = 1, \ldots, n; \ t = 1, \ldots, T-1; \ s \in S_t)$$

where $\overline{\mu}_{jt}]$ denotes expected rate of return of $\mu_{jt}^{(i)}$, $\sigma_{jt}]$ denotes standard deviation of rate of return of $\mu_{jt}^{(i)}$, and σ_{jt}^S denotes standard deviation of $\overline{\mu}_{jt}^s$. The mean and standard deviation of the scenario tree model match those of the hybrid model. However, the correlations between assets and the serial correlations of the scenario tree model does not match those of the hybrid model. This is because of the technical issue.

③ Price ρ_{jt}^s are calculated as follows;

$$\rho_{j1}^s = \rho_{j0}(1 + \mu_{j1}^s), \ (j = 1, \ldots, n; \ s \in S_1)$$

$$\rho_{jt}^s = \rho_{j,t-1}^{s'}(1 + \mu_{jt}^s), \ (j = 1, \ldots, n; \ t = 2, \ldots, T-1; \ s \in S_t)$$

$$\rho_{jT}^{(i)} = \rho_{j,T-1}^{s'}(1 + \mu_{jT}^{(i)}), \ (j = 1, \ldots, n; \ i \in V_{T-1}^{s'}; \ s' \in S_{T-1})$$

The number of scenarios are also I in the scenario tree model. The number of paths through the node s' of time $T-1$ is $|V_{T-1}^{s'}|$ ($s' \in S_{T-1}$), which depends on the branching tree.

Step 3: Solving the scenario tree model, and calculating optimal investment proportions

We can derive optimal investment units by solving the scenario tree model; z_{j0}^*, z_{jt}^{s*}(optimal solutions of risky asset) and v_0^*, v_t^{s*} (optimal solutions of cash). Optimal investment proportions are computed as follows;

$w_{j0}^* \equiv \frac{\rho_{j0} z_{j0}^*}{W_0}$: Investment proportion of risky asset j at time 0.

$c_0^* \equiv \frac{v_0^*}{W_0}$: Investment proportion of cash at time 0.

$w_{jt}^{s*} \equiv \frac{\rho_{jt}^s z_{jt}^{s*}}{W_t^{s*}}$: Investment proportion of risky asset j for time t and state s.

$c_t^{s*} \equiv \frac{v_t^{s*}}{W_t^{s*}}$: Investment proportion of cash for time t and state s.

Step 4: Evaluation of optimal solutions of the scenario tree model on the simulated paths

We evaluate the optimal investment proportions derived from the scenario tree model on the simulated paths. The expected terminal wealth and risk(LPM$_1$) are calculated using rates of return on simulated paths to describe the relationship curve between them.

$$\text{Expected terminal wealth} : \overline{W}_T \equiv \frac{1}{I} \sum_{i=1}^{I} W_T^{(i)*}$$

$$\text{Risk} : LPM_1 \equiv \frac{1}{I} \sum_{i=1}^{I} \max \left(W_G - W_T^{(i)*}, 0 \right)$$

where

$$R_1^{(i)*} = \sum_{j=1}^{n} \left(1 + \mu_{j1}^{(i)} \right) w_{j0}^* + (1 + r_0) c_0^*, \ (i \in V_1^s; \ s \in S_1)$$

$$R_t^{(i)*} = \sum_{j=1}^{n} \left(1 + \mu_{jt}^{(i)} \right) w_{j,t-1}^{s'*} + \left(1 + r_{t-1}^{(i)} \right) c_{t-1}^{s'*}, \ (i \in V_t^s; \ t = 2, \ldots, T; \ s' \in S_{t-1})$$

$$W_T^{(i)*} = \left(\prod_{t=1}^{T} R_t^{(i)*} \right) W_0, \ (i = 1, \dots, I)$$

4 Numerical tests : comparison of the models

We report results of numerical tests [12] . We compare two models numerically; the hybrid model with the fixed-proportion strategy and the scenario tree model. In addition, the hybrid model with the fixed-unit strategy is also tested for the purpose of reference. Four assets(stock, bond, convertible bond(CB), and cash) are solved over four periods. The number of scenarios(simulated paths) is 10,000. The number of constraints except non-negative constraints is about 50,000, and the number of decision variables is also about 50,000. The size of branching tree depends on the case. .

Initial prices of stock, bond, and CB can be assumed to be 1 without loss of generality. The initial call rate is 0.44%. The initial wealth is 100 million Japanese yen, and the target terminal wealth is also 100 million Japanese yen.

We have four kinds of numerical tests;

<u>Case A1</u>: Basic results for the 5-4-3 branching tree using statistics of historical data.

<u>Case B1</u>: Basic results for the 5-4-3 branching tree using virtual statistics considering various serial correlations.

<u>Case A2</u>: Comparison of the results for various numbers of N-N-N branching trees using statistics of historical data. (N= $2, 3, \dots, 13$)

<u>Case A3</u>: Comparison of the results for various branching trees under the same number of nodes at time 3 using statistics of historical data.

We examine the basic features about the difference between the scenario tree model and the hybrid model in the Case A1. The number of states(paths) which comes out of each state(node) at time $T-1$ is 166 or 167 in the scenario tree model(hybrid model). We test how the serial correlations affect the difference between two kinds of models in the Case B1. The larger the size of the branching tree, the larger the number of states from time 1 to time 3 in the scenario tree model. On the other hand, the number of states(paths) remains in the hybrid model even if the size of the branching tree become larger. How does this difference between two kinds of models affect the results? We answer this question in the Case A2. It is important to compare models under the various structures of branching trees. In the Case A3, we examine the difference of these models under the condition that the number of decision nodes(states) remains at time 3 [13].

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[12] All of the problems are solved using NUOPT(Ver. 5.1.0a) — mathematical programming software developed by Mathematical System, Inc. on Windows 2000 personal computer which has 1.8 GHz CPU and 768 MB memory.

[13] The number of states(paths) which comes out of each state(node) at time $T-1$ depends on 'N-N-N' in the Case A2, but they have the same value in the Case A3.

Legend symbols in the figures of efficient frontier indicate the following meaning.

Scenario	Efficient frontier of the scenario tree model
Scenario(H)	Relationship curve between the expected terminal wealth and risk of the scenario tree model evaluated on simulated paths
Hybrid(R)	Efficient frontier of the hybrid model with the fixed-proportion strategy
Hybrid(U)	Efficient frontier of the hybrid model with the fixed-unit strategy

4.1 Case A1: Basic results for the 5-4-3 branching tree using statistics of historical data
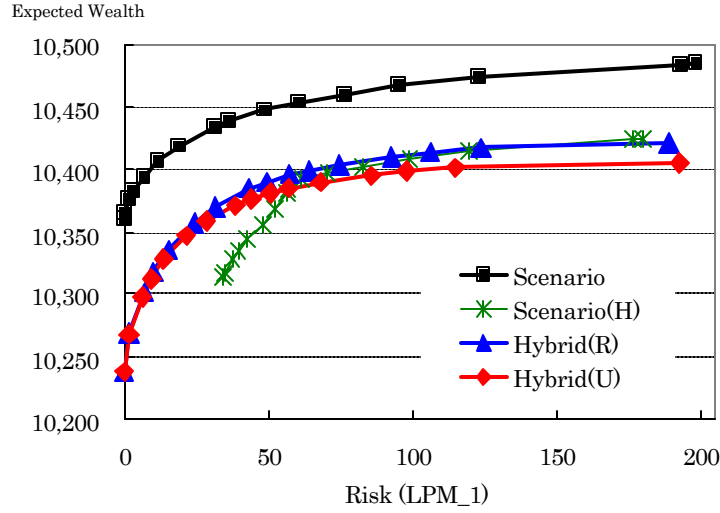
Figure 4: Efficient Frontier

Figure 4 shows the efficient frontier of the 5-4-3 branching tree using statistics of historical data. The problem solved by the scenario tree model is over-evaluated because of the insufficient description of uncertainties associated with asset returns. When we evaluate optimal solutions of the scenario tree model on the simulated paths, the efficient frontier moves downwards, and it cannot have low risk. The efficient frontier of the hybrid model with the fixed-proportion strategy['Hybrid(R)'] is better than the relationship curve of the scenario tree model evaluated on the simulated paths['Scenario(H)']. The hybrid model can evaluate and control risk better than the scenario tree model. The fixed-proportion strategy dominates the fixed-unit strategy in the hybrid model. This is because we need to hold cash after time 1 to execute transactions for the fixed-unit strategy in the simulated path approach. On the other hand, we do not always have to hold cash for the fixed-proportion strategy. When γ is large, two strategies have almost the same values, because cash is held to reduce risk. When γ is small, the efficient frontier of the hybrid model with the fixed-unit strategy['Hybrid(U)'] is worse than the relationship curve of the scenario tree model evaluated on the simulated paths['Scenario(H)'] due to the same reason.

We can verify by studying Figure 5, which shows the average investment ratios at each time. The horizontal axis is 16 kinds of the risk-averse coefficients(γ). The smaller the risk-averse coefficients(γ), the more cash the fixed-unit strategy holds than the fixed-proportion strategy.

Dynamic portfolios of two strategies of the hybrid model are similar each other except cash.

Optimal solutions of the scenario tree model are different from those of the hybrid model, and they are very extreme solutions at time 0. This is because the scenario tree model has less scenarios in period 1 than those of the hybrid model.
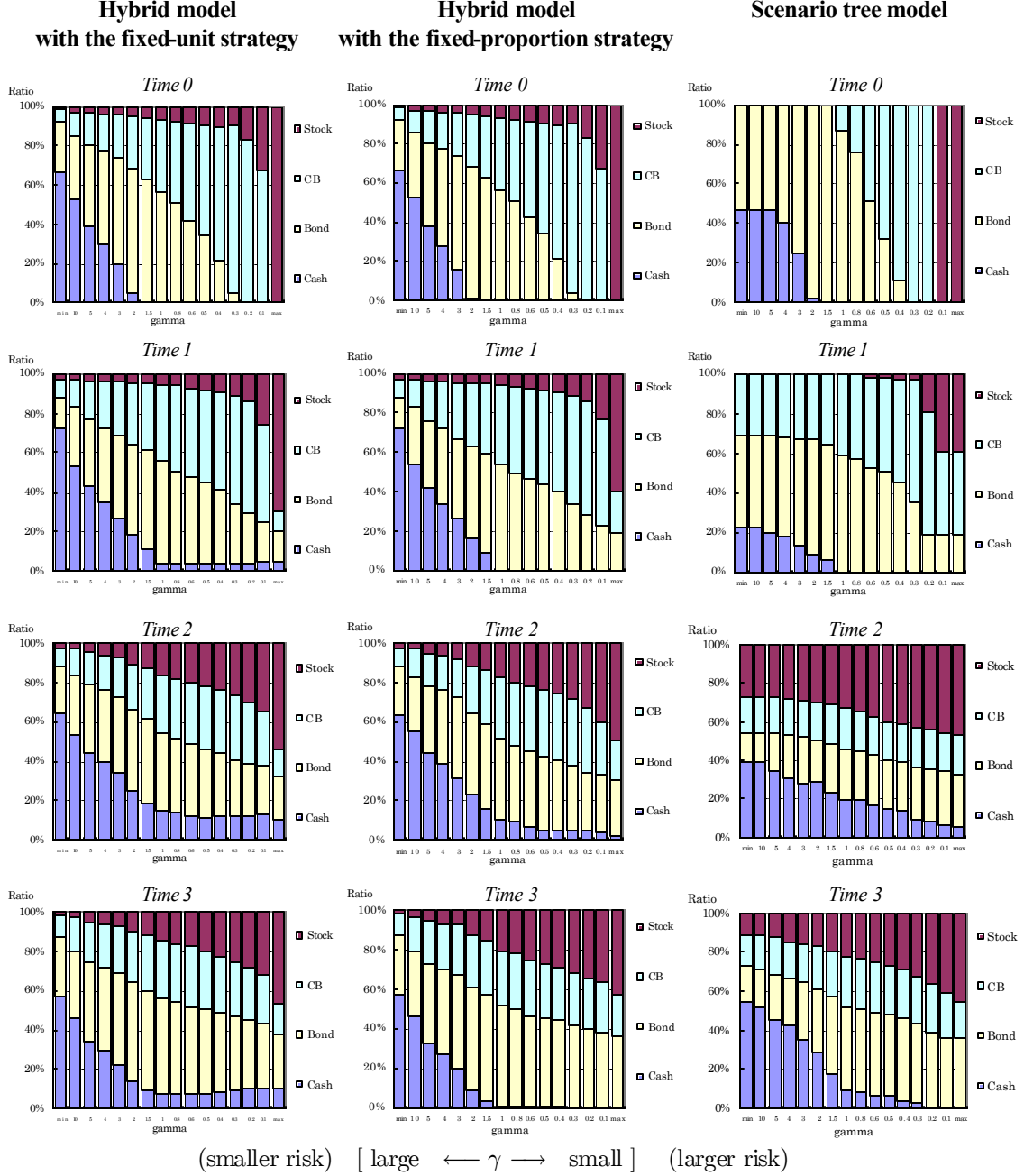

| | Hybrid model with the fixed-unit strategy | Hybrid model with the fixed-proportion strategy | Scenario tree model |

(smaller risk)　[large　$\longleftarrow \gamma \longrightarrow$　small]　(larger risk)

Figure 5: Average Investment Ratio

4.2　Case B1: Basic results for the 5-4-3 branching tree using virtual statistics considering the serial correlation

Figure 6 shows 11 kinds of the efficient frontiers for the 5-4-3 branching tree using virtual statistics considering various serial correlations. The larger the parameter c, the closer the

efficient frontier of the hybrid model['Hybrid(R)'] to that of the relationship curve of the scenario tree model evaluated on the simulated paths['Scenario(H)'], but which cannot also have the solutions of low risk in this case. The smaller the absolute value of the parameter c is, the closer the efficient frontier of the scenario tree model is to both the efficient frontier of the hybrid model['Hybrid(R)'] and the relationship curve of the scenario tree model evaluated on the simulated paths['Scenario(H)']. In other words, the larger the absolute value of the parameter c, the more over-evaluated the efficient frontier of the scenario tree model['Scenario'].
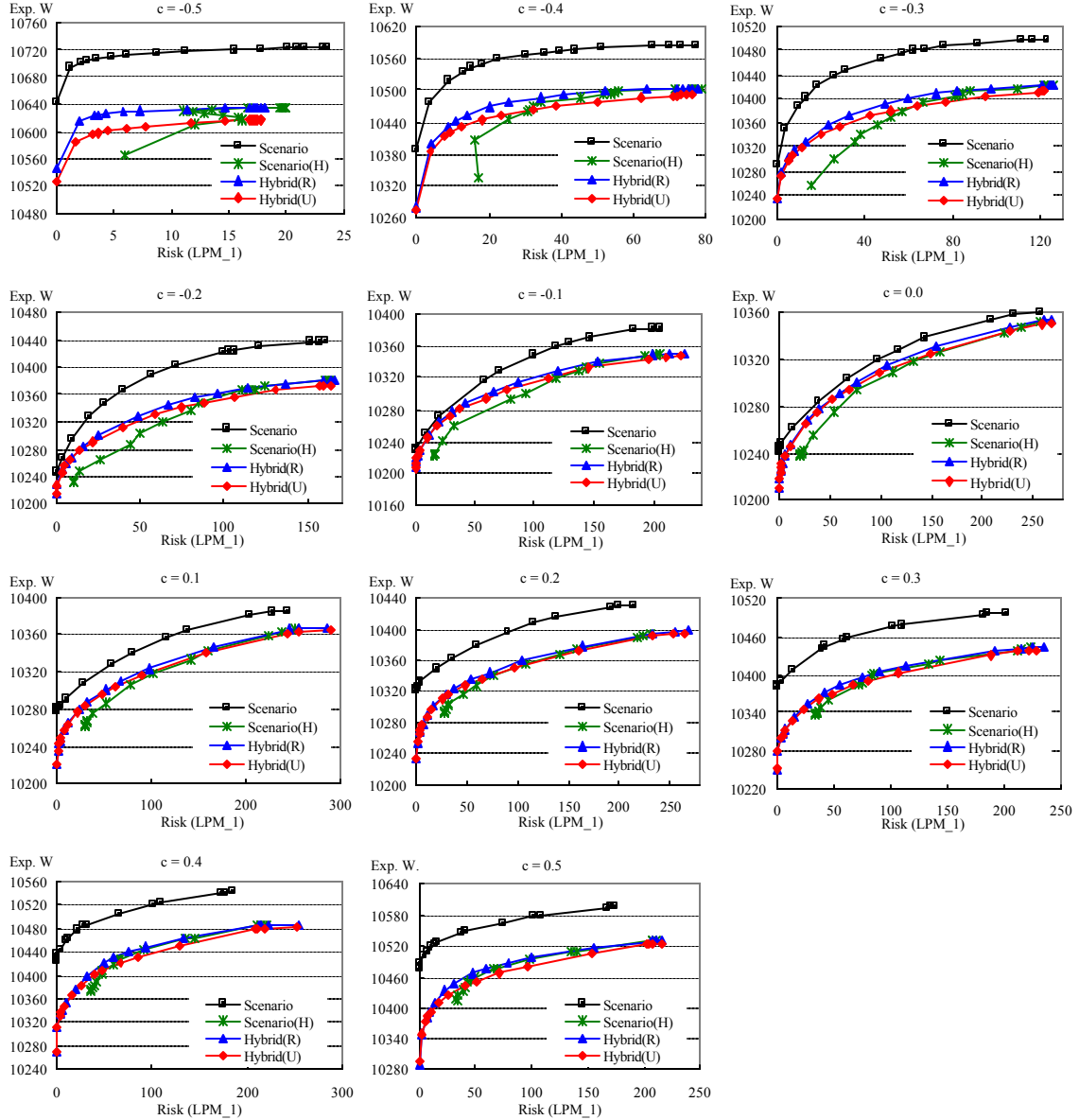
Figure 6: Efficient frontier

Figure 7 and Figure 8 show the investment ratios at time 0 for 16 kinds of risk-averse coefficients (γ). The horizontal axis of Figure 7 is the risk averse coefficients (γ), and the horizontal axis of Figure 8 is the serial correlation parameter (c).
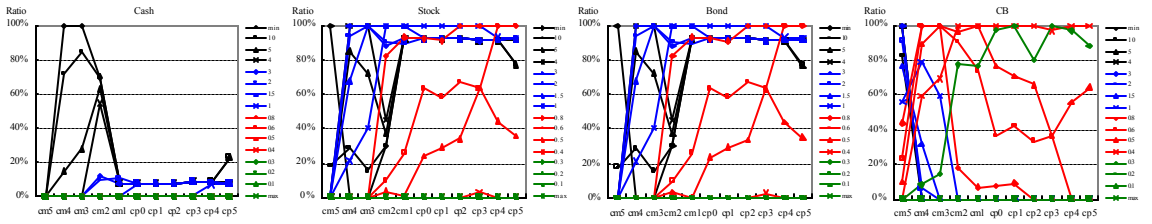
Hybrid model with the fixed-unit strategy   Hybrid model with the fixed-proportion strategy   Scenario tree model

(smaller risk)   [large   ⟵ γ ⟶   small]   (larger risk)
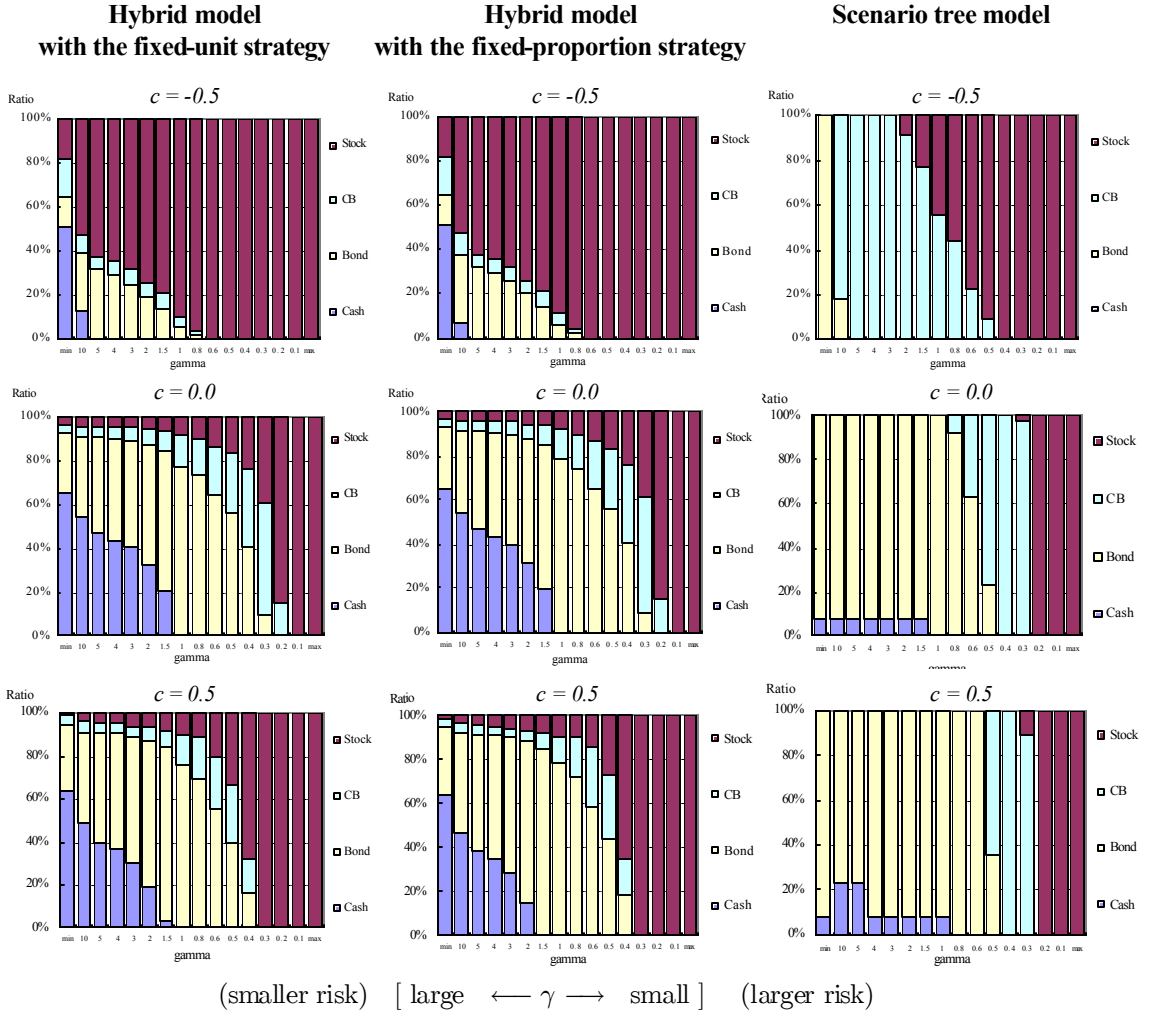
Figure 7: Investment Ratio at time 0

Hybrid model with the fixed-proportion strategy
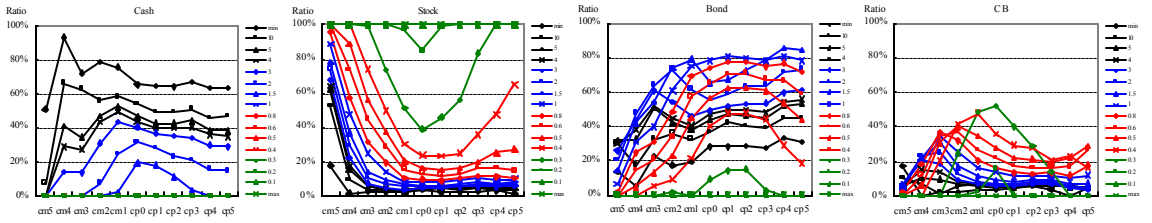
Scenario tree model

(negative correlation)   [small   ⟵ c ⟶   large]   (positive correlation)

Figure 8: Investment Ratio at time 0

Whatever the parameter c is set, the optimal solutions between the hybrid model and the scenario tree model are different as well as the result of the Case A1. The optimal solutions of two strategies of the hybrid model are similar as well. We find the relationship between the parameter c and the investment proportions in the hybrid model in Figure 8. However, we cannot find the relationship in the scenario tree model. This is because the original serial correlations do not seem to be kept when we construct the scenario tree from the simulated paths. If we examine the relationship in the scenario tree model, we need to develop the other procedure to make a scenario tree. This issue will be our future research.

## 4.3 Case A2: Comparison of the N-N-N branching trees using statistics of historical data
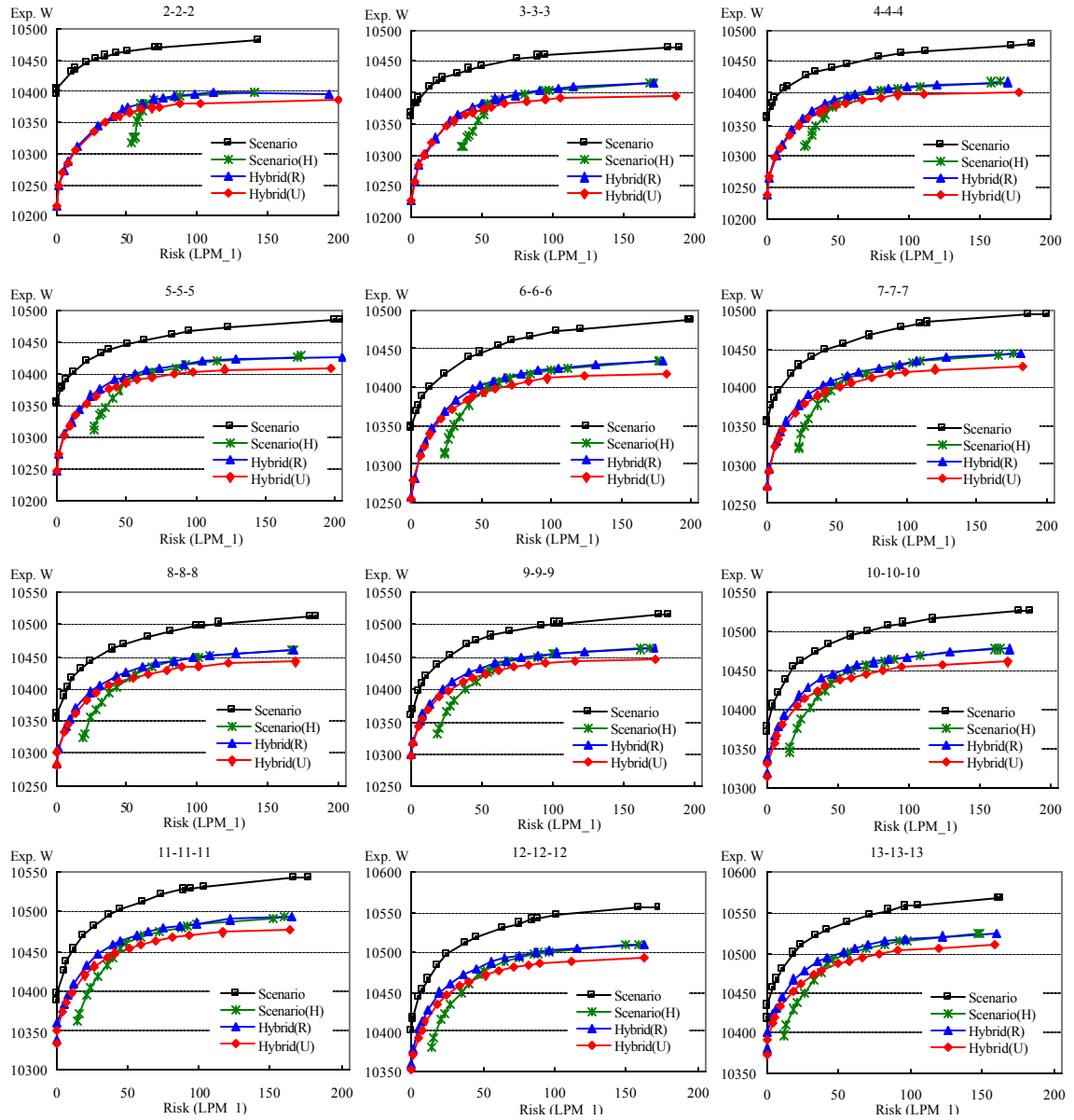
Figure 9: Efficient frontier

Figures 9 shows the efficient frontiers for various numbers of branching trees (N-N-N branching trees) using statistics of historical data. Even if the number of branching trees increases, the degree of difference between two models is similar to that of the 5-4-3 branching tree. The efficient frontier derived from the scenario tree model is also over-evaluated. But the larger the size of the branching tree, not only the better the relationship curve but also the smaller the minimum risk of the relationship curve of the scenario tree model evaluated on the simulated paths['Scenario(H)']. This reason is that the scenario tree model that has the larger branching tree can describe more accuracy of uncertainties, and control risk. However, the relationship curve is still dominated by the efficient frontier of the hybrid model with the fixed-proportion strategy.

Hybrid model with the fixed-proportion strategy
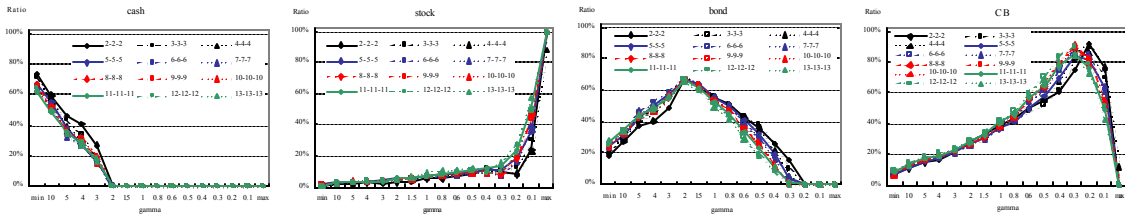
Scenario tree model
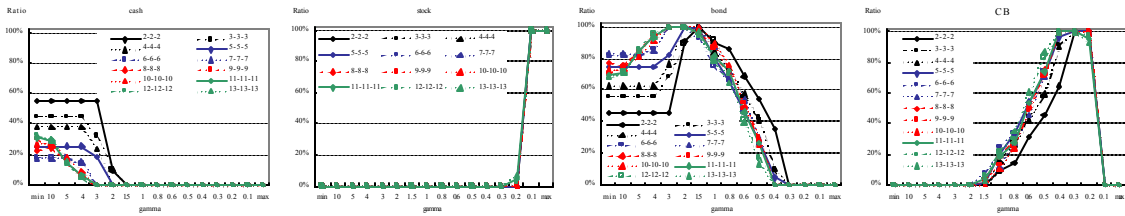
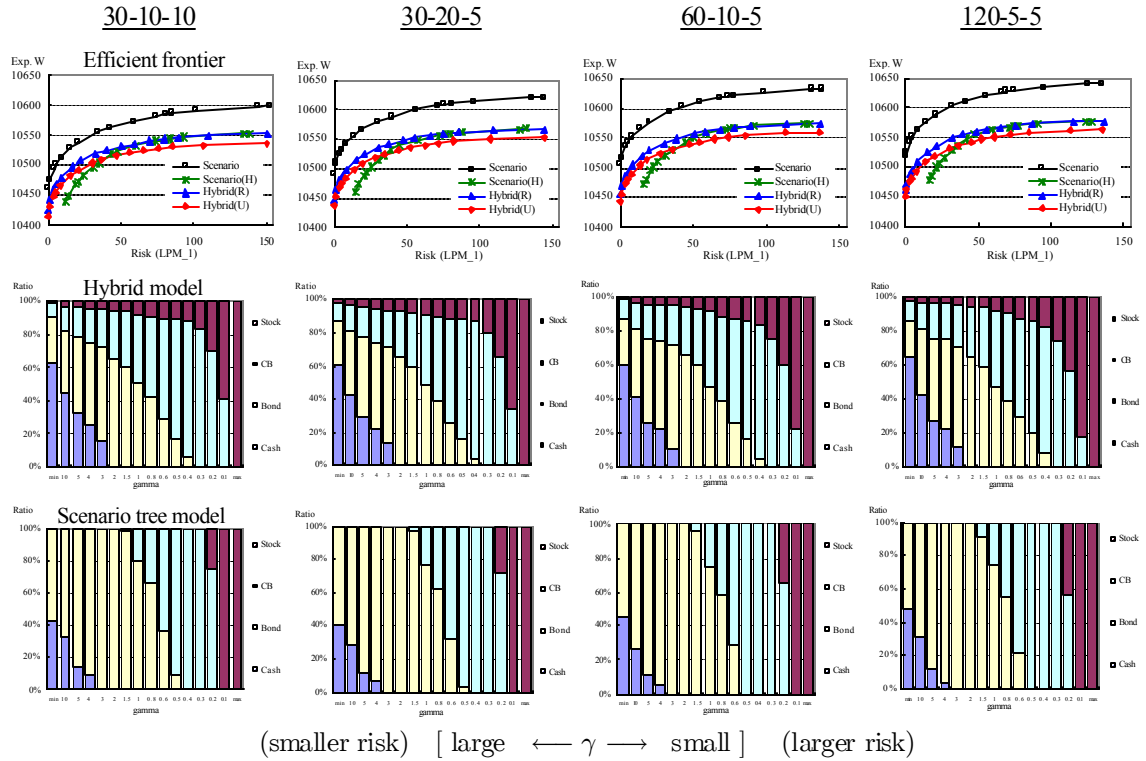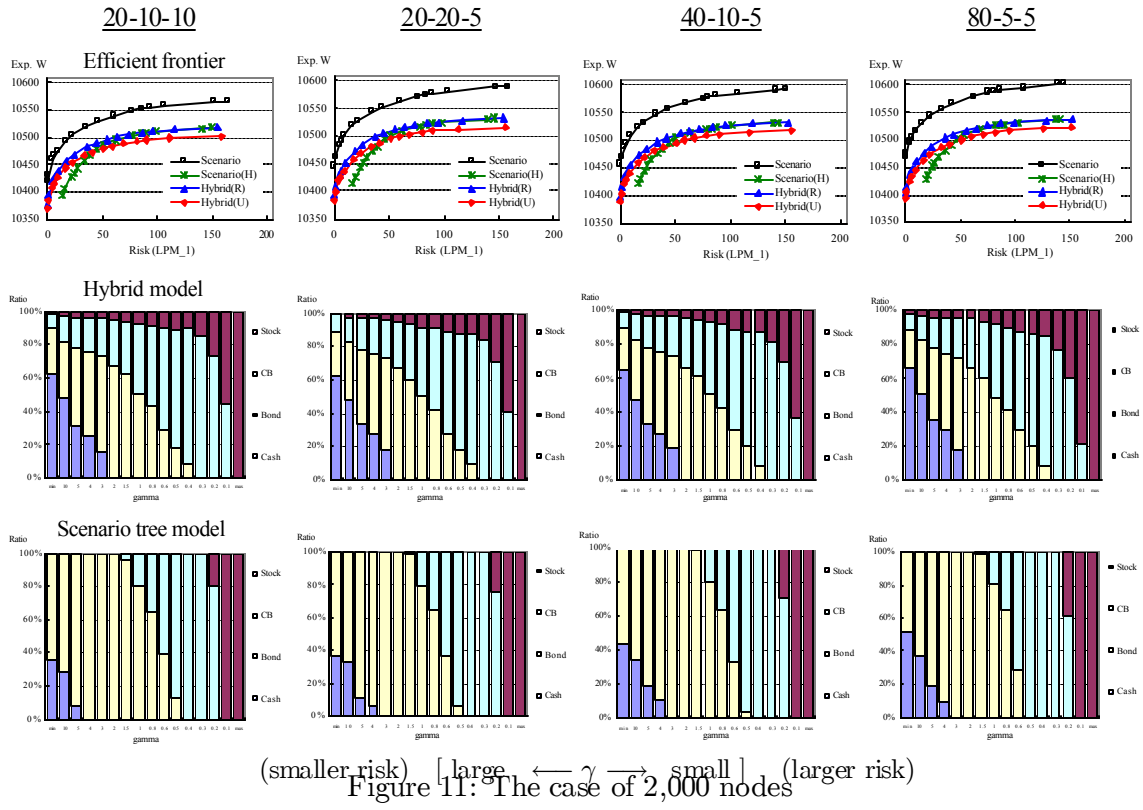(smaller risk)　[large　⟵ γ ⟶　small]　(larger risk)

Figure 10: Investment Ratio at time 0

Figure 10 show the investment ratios at time 0 of the two models. Optimal solutions of the scenario tree model are different from those of the hybrid model like the case of the 5-4-3 branching tree. Optimal solutions of the scenario tree model are more sensitive to the change of the number of branching tree than those of the hybrid model. The larger the size of the branching tree, the more risky assets we tend to invest in at time 0 for both models. This is because more flexible investments can control risk, even if more risky assets are invested in at time 0.

4.4　Case A3 : Comparison of the branching trees under the same number of nodes at time 3 using statistics of historical data

Figures 11 and 12 show the efficient frontiers and the investment ratios at time 0 of the two models. The larger the number of nodes, the more upwards the efficient frontier moves. However, the relationship between the two models is almost the same results as the previous cases.

20-10-10   20-20-5   40-10-5   80-5-5

(smaller risk) [large ← γ → small] (larger risk)

Figure 11: The case of 2,000 nodes

30-10-10   30-20-5   60-10-5   120-5-5

(smaller risk) [large ← γ → small] (larger risk)

Figure 12: The case of 3,000 nodes

We examine how it differs by the difference of branching trees for each model. Efficient frontiers can be divided into some groups for every number of nodes as follows. The figure in the parenthesis expresses the number of nodes at time 2.

# of nodes	branching trees			
2,000	20-10-10(200)	20-20-5(400)	40-10-5(400)	80-5-5(400)
3,000	30-10-10(300)	30-20-5(600)	60-10-5(600)	120-5-5(600)

We can compare these branching trees as the following manner, and group pairs of branching trees.

Group 1: Comparison of the cases that the numbers of nodes at both time 1 and time 2 are different each other, such as '20-10-10'(20 at time 1 and 200 at time 2) and '40-10-10'(40 at time 1 and 400 at time 2).

Group 2: Comparison of the cases that the numbers of nodes are same at time 1, but are different each other at time 2, such as '20-10-10'(20 at time 1 and 200 at time 2) and '20-20-5'(20 at time 1 and 400 at time 2).

Group 3: Comparison of the cases that the numbers of nodes are different each other at time 1, but are same at time 2, such as '20-20-10'(20 at time 1 and 400 at time 2) and '40-10-10'(40 at time 1 and 400 at time 2).

The pairs of efficient frontiers in Group 1 are the most different each other, and so are they in Group 2. The pairs of efficient frontiers in Group 3 are more similar than those of the other groups relatively. This is because there are a lot of paths through each node at the time 1. The difference seems to be reduced in comparison with the case that the number of paths are different each other at time 2 as in Group 2.

5   Concluding Remarks

The scenario tree model is typically used for the dynamic portfolio optimization, but it has a serious problem about describing the uncertainties. The hybrid optimization model using simulated paths and the decision tree allows both the describing of the uncertainties with high accuracy and the making of conditional decisions. The previous papers(Hibiki, 2001b and 2003) show some features of the hybrid model, but do not examine how the hybrid model is better than the scenario tree model. In this paper, we compare two types of multi-period stochastic optimization (MPSO) models by using numerical tests, and illustrate the difference between them. Our contributions and related future research of this paper are as follows;

(1) We develop the iterative algorithm to solve the hybrid model with the fixed-proportion strategy approximately. This algorithm is applied to the non-convex problem, but it is much stable because the linear programming problem is solved iteratively. This algorithm seems to work well, but we must keep considering the new algorithm because it does not guarantee to derive the global optimal solutions.

(2) We develop the method of comparing two kinds of models with the different scenario structures. In this method, the mean and standard deviation of asset returns of the simulated paths match those of the scenario tree model each period, but the correlations do not match each other because of the technical issue. We need to develop the correlaton matching method to construct a scenario tree from simulated paths, though it is a difficult problem. Moreover, we need to compare two models using the 'tree to path' procedure in order to compare it with the 'path to tree' procedure we adopt in this paper, though it is difficult to construct the appropriate scenario tree.

(3) We test some cases to compare two kinds of models using numerical examples. We can show that the hybrid model can evaluate and control risk better than the scenario tree model by using some numerical tests. Two kinds of summary statistics are tested under the scenario generating procedure. We should compare them under the various scenario generating models.

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